Trigonometric Substitution for Integration:

A very useful technique to evaluate integrals in calculus is "trig substitution". Trig substitution is used when we have a radical in the integrand and the radicand is of second degree. Look at the integral below: $\int\sqrt{x^2-9}dx$. On first glance, it is tempting to write the radicand as $(x^2-9)^{\frac{1}{2}}$ and try to use the technique $\int u^n du = \frac{u^{n+1}}{n+1}+C$ where n = ½. A problem occurs here: $u=(x^2-9)$ and $du=2xdx$. We are missing the 2 and the "x" in the above integral. An "x" cannot be produced here, so we are forced to use another technique.

The key to trig substitution is to remember the integration formulas for the inverse trigonometric functions. They are illustrated below. Remember, **u** = a function of x and "**a**" = a constant:

1) $\int\frac{du}{\sqrt{a^2-u^2}}=\sin^{-1}\frac{u}{a}+C$

2) $\int\frac{du}{a^2+u^2}=\frac{1}{a}\tan^{-1}\frac{u}{a}+C$

3) $\int\frac{du}{|u|\sqrt{u^2-a^2}}=\frac{1}{a}\sec^{-1}\frac{u}{a}+C$

To use trig substitution on $\int\sqrt{x^2-9}dx$ or any other integral with a radical in the integrand and the radicand is of second degree, we ask ourselves, "**Which one of the above formulas does** $\int\sqrt{x^2-9}dx$ **look like?"** The integral $\int\sqrt{x^2-9}dx$ "looks like formula #3" because u = x is out in front in the radical and a^2 appears to be 9. Hence, a = 3. Use the substitution $x=a\sec\theta$ because formula #3 deals with the inverse secant function. Hence, $x=3\sec\theta$ and $dx=3\sec\theta\tan\theta d\theta$. Substituting into $\int\sqrt{x^2-9}dx$, we get $3\int\sqrt{9\sec^2\theta-9}\ \sec\theta\tan\theta d\theta=3\int\sqrt{9(\sec^2\theta-1)}\ \sec\theta\tan\theta d\theta$. Using the trigonometric identity $\mathbf{\tan^2\theta+1=\sec^2\theta}$, we get $\mathbf{\tan^2\theta=\sec^2\theta-1}$ and $3\int\sqrt{9\tan^2\theta}\ \sec\theta\tan\theta d\theta$. The integral = $9\int\tan^2\theta\sec\theta\, d\theta$.

We must use integration by parts on the above integral; i.e., $\int u\,dv=uv-\int v\,du$. Letting $dv=\tan\theta\cdot\sec\theta d\theta$, we get $v=\sec\theta$. Letting $u=\tan\theta$, we get $du=\sec^2\theta d\theta$(*The 9 from the above integral is omitted for the moment.)*. Hence, $\int\tan^2\theta\sec\theta d\theta=\sec\theta\tan\theta-\int\sec^3\theta d\theta^{**}$. Using the trigonometric identity $\mathbf{\tan^2\theta+1=\sec^2\theta}$ on the second integral of **, we get $\int\tan^2\theta\sec\theta d\theta=\sec\theta\tan\theta-\int(\tan^2\theta+1)\sec\theta d\theta.^{**}$

Doing algebra on the above equation**, we have

$2\int \tan^2\theta\sec\theta d\theta = \sec\theta\tan\theta - \int \sec\theta d\theta = \sec\theta\tan\theta - \ln|\sec\theta + \tan\theta| + C.$ Hence,

$9\int \tan^2\theta\sec\theta\, d\theta = \frac{9}{2}\sec\theta\tan\theta - \frac{9}{2}\ln|\sec\theta + \tan\theta| + C$ = the original integral.

At this point, it looks as though we are finished. WE ARE NOT FINISHED because the original variable of integration is "x" not **θ**! We must convert back to x. Notice that $x = 3\sec\theta$ or $\sec\theta = \frac{x}{3}$. Hence, $\cos\theta = \frac{3}{x}$. We must solve for sin**θ** and tan**θ** by setting the triangle below:

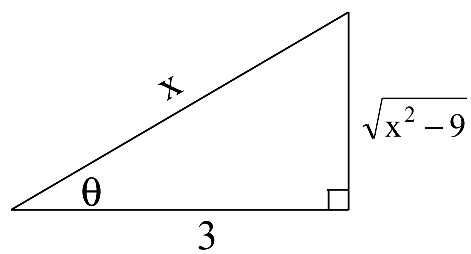

Figure 1: This is the triangle that we have to use to figure out the other trig functions.

Using the Pythagorean theorem, we have the opposite side of **θ** in the above triangle $= \sqrt{x^2+9}$. Hence, $\sin\theta = \frac{\sqrt{x^2-9}}{x}$ and $\tan\theta = \frac{\sqrt{x^2-9}}{3}$. Substituting, we have

$$\int \sqrt{x^2-9}\, dx = \frac{9}{2}\frac{x}{3}\left(\frac{\sqrt{x^2-9}}{3}\right) - \frac{9}{2}\ln\left|\frac{x}{3} + \frac{\sqrt{x^2-9}}{3}\right| + C = \frac{x\sqrt{x^2-9}}{2} - \frac{9}{2}\ln\left|\frac{x}{3} + \frac{\sqrt{x^2-9}}{3}\right| + C$$