SUGGESTED GUIDELINES FOR MAX-MIN PROBLEMS IN CALCULUS:

The main problem in max-min problems is setting up the equation to be differentiated. Max-Min problems answer the question: What value **a** will maximize/minimize something? There is no formula to solve them. However, the below guidelines are applicable to every max-min problem:

i) Set up the equation **F** to be differentiated. Use formulas that were learned in previous courses. Many high school geometry formulas that deal with area, surface area, or volume may need to be used.

ii) Set up any constraint equation(s) **G** that arise in the problem if **F** has two or more variables in it. If **F** has one variable, go directly to step **iv**).

iii) Substitute equation(s) **G** into **F** so that **F** has one variable in it. Remember, you cannot take the derivative of an equation with two or more variables in it. College freshman and AP calculus courses do not allow this.

iv) Take the first derivative of $F = F'(x)$. Set $F'(x) = \mathbf{0}$ and solve for x. Say $F'(\mathbf{a}) = \mathbf{0}$.

v) Take the second derivative of $F = F''(x)$. Substitute each value $x = \mathbf{a}$ obtained in step **iv**) into $F''(x)$. If $F''(\mathbf{a}) > \mathbf{0}$, then F(x) has a **minimum** at $x = \mathbf{a}$. If $F''(\mathbf{a}) < \mathbf{0}$, then F(x) has a **maximum** at $x = \mathbf{a}$. If $F''(\mathbf{a}) = \mathbf{0}$, use the first derivative test to determine if $x = \mathbf{a}$ yields a maximum or minimum.

vi) After solving for the values $x = \mathbf{a}$, REREAD THE PROBLEM! $x = \mathbf{a}$ may not be the value the author, teacher, or professor wants. You will have to use the value $x = \mathbf{a}$ to get the wanted values in the problem. If you do not solve for what is wanted, you will get the problem wrong even though you solved it correctly.

The next two examples will illustrate how to apply the above guidelines:

Example 1: Suppose that you buy **36** feet of fencing. What are the dimensions of the rectangular plot of maximum area?

Solution: It is a good idea to provide a rough drawing to understand the problem further. The drawing is below.

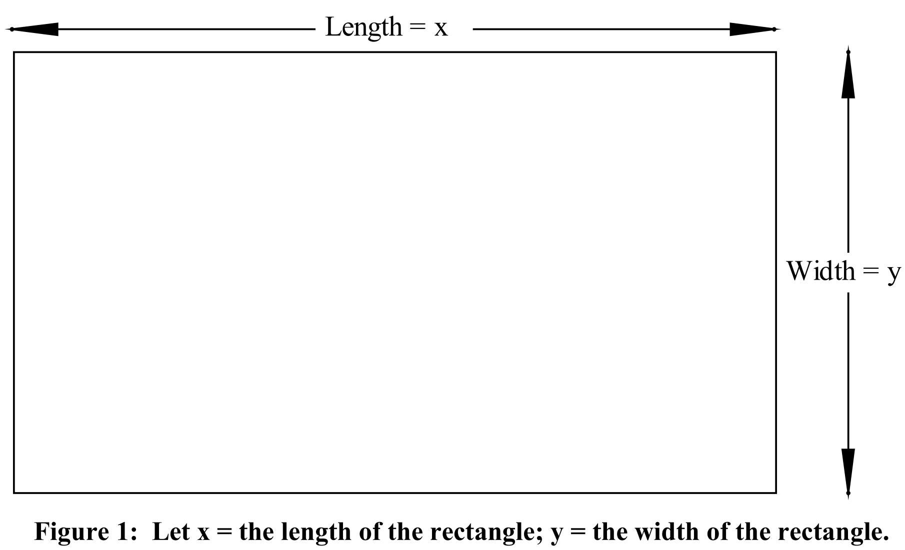

Figure 1: Let x = the length of the rectangle; y = the width of the rectangle.

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As illustrated in figure 1, **x** = the length of the rectangle, and **y** = the width of the rectangle. By the area formula, A = lw implies **A = xy**. Since we are maximizing area, **A = xy** is the equation to differentiated. Hence, we have finished step **i**) above.

Notice that **A = xy** is an equation with two variables "**x**" and "**y**" in it. We need another equation to reduce the number of variables to one. The next equation is the constraint equation that needs to be done in step **ii**) above. In the problem above, we had **36** feet of fencing. What this tells us is the perimeter of the rectangle equals **36**; i.e., **2x + 2y = 36**.

We must do step **iii)** above. Solving for y, we get **y = 18 – x**. Substituting **y = 18 – x** into **A = xy**, we have **A = x(18 – x)**. Notice that A has one variable in it, so we can differentiate it.

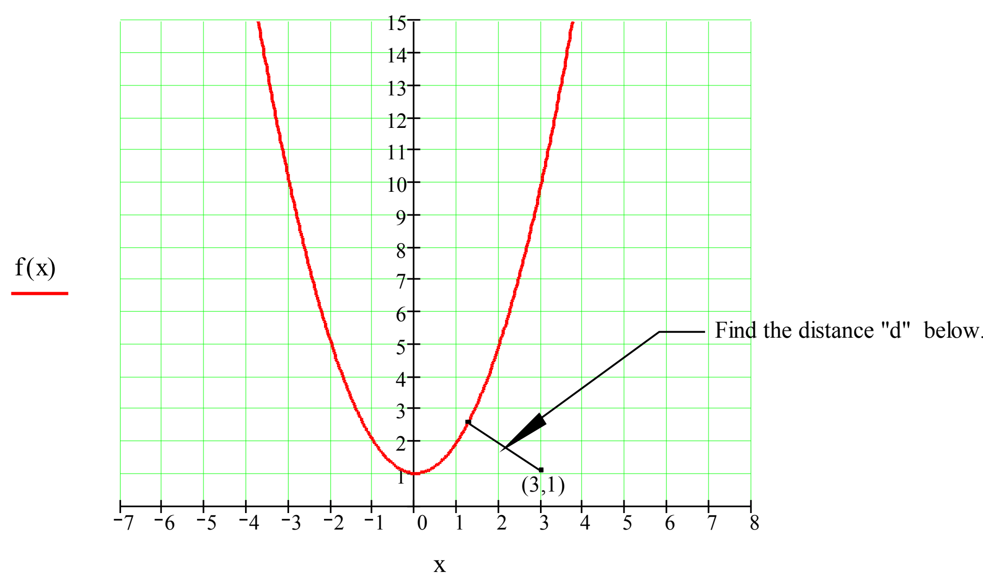
Going to step **iv)**, we differentiate the equation $\mathbf{A = x(18 - x)} = \mathbf{18x - x^2}$ obtaining $\frac{dA}{dx} = 18 - 2x$. Setting the derivative = **0** and solving we get $\mathbf{18 - 2x} = \mathbf{0} \Rightarrow \mathbf{x = 9}$.

Going to step **v)**, we take the second derivative of **A = x(18 – x)** obtaining $\frac{d^2A}{dx^2} = -2$. Notice that the second derivative is negative for all values of **x**. Therefore, we have maximum at **x = 9**.

At this point, it is tempting to stop. REREAD THE PROBLEM!(step **vi**)) The problem said to find the dimensions of the rectangle. **x = 9** gives us only the length of the rectangle. We must find the width of the rectangle. Substituting into the constraint equation above, we have **y = 18 – 9** = **9**. Hence, we have a square. We would never have known this had step **vi**) not been done.

Example 2: Find the point on the graph $f(x) = x^2 + 1$ that is closest to the point (3, 1). The graph is illustrated below(Figure 2):

$f(x) := x^2 + 1$



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Solution: We must use the distance formula to describe the distance "d" from (3, 1) to $f(x) = x^2 + 1$. Applying the distance formula, we have $d(x, y) = \sqrt{(x-3)^2 + (y-1)^2}$. (Step **i**))

Once again, we have an equation in two variables that we cannot differentiate. We must go to step **ii**) to find another equation or relationship to reduce the number of variables to one.(*Constraint equation(s)*) Actually, $f(x) = y = x^2 + 1$ is a constraint equation we can use in the distance formula above.

Going on to step **iii**) and substituting, we have $d(x, y) = \sqrt{(x-3)^2 + (x^2 + 1 - 1)^2} =$ $d(x) = \sqrt{(x-3)^2 + (x^2)^2} = \sqrt{(x-3)^2 + x^4} = \sqrt{x^4 + x^2 - 6x + 9}.$

By step **iv**), we just differentiate d(x). However, taking the derivative of something with a radical in it is going to be cumbersome. We can minimize $D(x) = d(x)^2$ and get the same answer. This trick is not mentioned in many calculus books, but it works! Because $a \le b \Rightarrow a^2 \le b^2$ where a, b ≥ 0 and distance d(x) ≥ 0, the trick will work. Differentiating, we have $D(x) = x^4 + x^2 - 6x + 9 \Rightarrow D'(x) = 4x^3 + 2x - 6$. Setting $D'(x)$ equal to zero, we have $D'(x) = 4x^3 + 2x - 6 = 0 \Rightarrow 2x^3 + x - 3 = 0 \Rightarrow x = 1$. One can use synthetic division or another factoring technique to deduce that $2x^3 + x - 3 = (x-1)(2x^2 + 2x + 3)$. The other roots are complex, and complex numbers are not allowed in freshman or AP calculus courses.

Going on to step **v**) and taking the second derivative, we have $D(x) = x^4 + x^2 - 6x + 9 \Rightarrow D'(x) = 4x^3 + 2x - 6 \Rightarrow D''(x) = 12x^2 + 2 > 0$ for all values x. Hence, x = 1 will yield a minimum which is what we expected.

Once again, step **vi**) says <u>REREAD THE PROBLEM</u>! The problem says to find the point that gives the minimum distance, not the above value of x! Substituting into $f(x) = x^2 + 1$, we get y = 2. Thus, the point (1, 2) will give the minimum distance.
Just for the sake of curiosity, the minimum distance is
$d(1,2) = \sqrt{(x-3)^2 + (y-1)^2} = \sqrt{(1-3)^2 + (2-1)^2} = \sqrt{4+1} = \underline{\underline{\sqrt{5}}}.$

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